Analysis 1

19 March 2024

Functions

A **function** is any rule for giving outputs to certain inputs.

A **real function** is a function where every input and every output is a real number. *We will only use real functions in this class.*

Often a function is described by a formula, although technically it doesn't have to be.

- $f(x) = x^3 - \frac{1}{2}x + 7$
- $f(t) =$ the number of meters a car moving at 5 kph travels in t seconds

$f(t) = \frac{5000}{3600}t$, or $f(t) = \frac{25}{18}t$

Types of functions

A **polynomial** in the variable x is a function that can be described by an expression of the form

$$😀x^n + 🧐x^{n-1} + \cdots + 😂x^2 + 🥺x + 😐,$$

where $n \geq 0$ is an integer.

In this class we will only use coefficients and variables that are real numbers.

A **rational function** is one polynomial divided by another (the denominator should not be exactly $g(x) = 0$).

Example: $\dfrac{x^4 + 2x^3 + \sqrt{2}x - 8}{x - 7}$.

Types of functions

Assume a, b, c, d are constants.

- $f(x) = a\,x^b$ is a **power function**.
  - This includes $5x^{1/2}$, which is $5\sqrt{x}$.
- $f(x) = a\,b^x$ is an **exponential function**.
  - This includes $5 \cdot 3^{2x}$ because that is also $5 \cdot 9^x$.
- $f(x) = a\sin(bx + c) + d$ is a **trigonometric function**, and so are ... $\cos$ ... and some others.
- $f(x) = a\ln(bx + c) + d$ is a **logarithmic function**.
  - This includes $\log_2(x)$ because that is also $\frac{1}{\ln(2)}\ln(x) \approx 1.443\ln(x)$.

Questions:

- Is x^8 a power function?
- Is x^8 it a polynomial?

- Is $\frac{3}{x}$ a power function?
- Is $\frac{3}{x}$ a polynomial?

- Is x^x a power function?

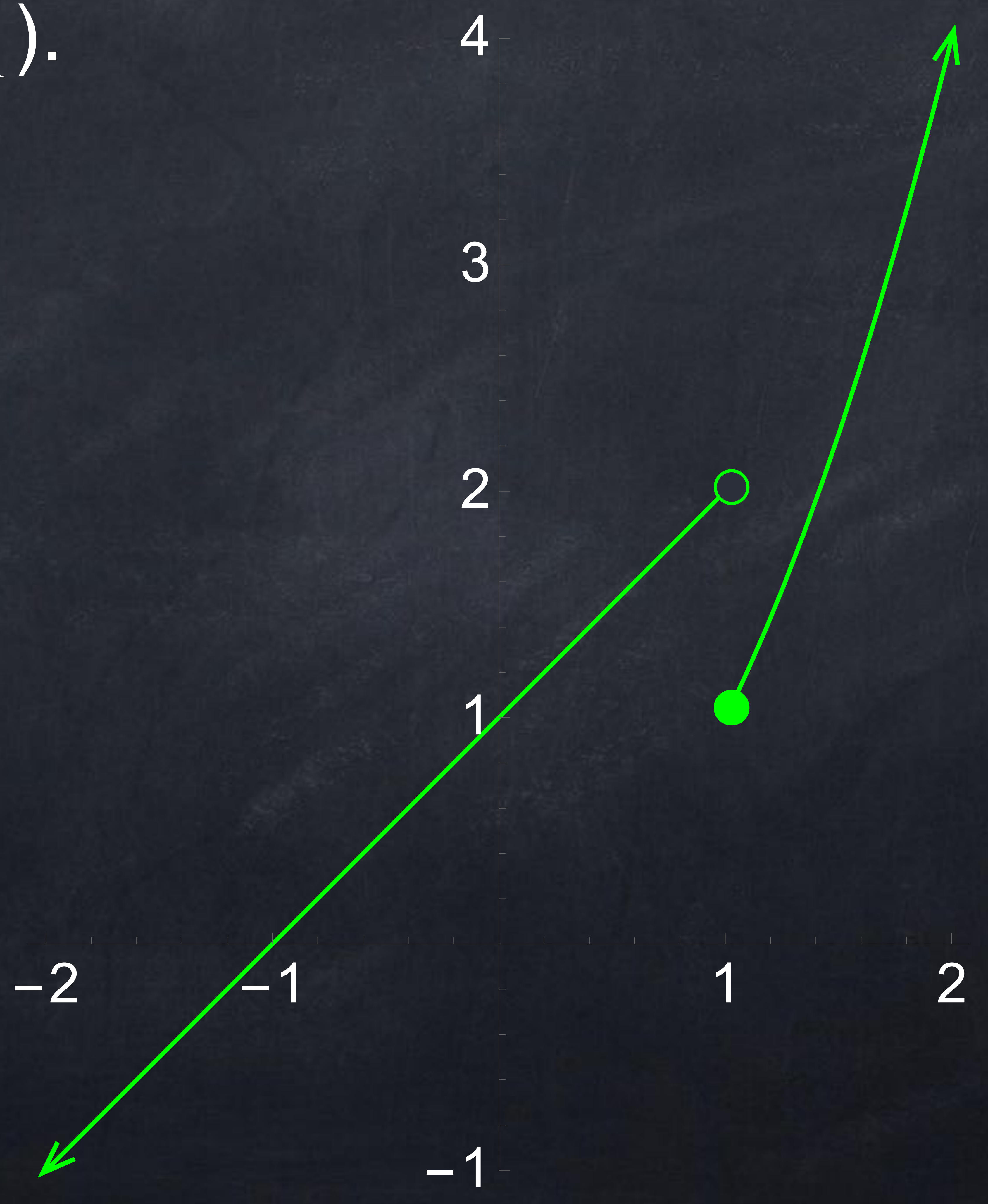
- Is 4^x a power function?

Piecewise functions

A **piecewise function** is one that uses different formulas for different inputs. We write these using a large "curly bracket" ($\{$).

Example: $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}.$

Note the "open circle" ○ at $(1,2)$ and the "filled circle" ● at $(1,1)$.



Piecewise functions

A **piecewise function** is one that uses different formulas for different inputs. We write these using a large “curly bracket” ({).

These kinds of functions are actually very common in the real world.

Domain

The **natural domain** (sometimes just called the **domain**) of a function $f(x)$ given by a formula is the largest possible subset of $\mathbb{R}$ on which the function is defined.

Examples:

- The natural domain of $\sqrt{x}$ is $[0, \infty)$, also written $\{x \in \mathbb{R} : x \geq 0\}$.
- The natural domain of $\sqrt{x+6}$ is $[-6, \infty)$.
- The natural domain of $\sqrt{x^2+6}$ is $\mathbb{R}$ (you can plug in *any* real x!).

To find the natural domain of a function, start with all of $\mathbb{R}$ and then remove numbers that would cause a “problem” in the formula.

There are three main kinds of problems:

- division by zero

$$f(x) = \frac{1}{x} \quad \rightarrow \quad x \neq 0$$

- even roots of negative numbers

$$f(x) = \sqrt{x} \quad \rightarrow \quad x \geq 0$$

$$f(x) = \sqrt[3]{x} \quad \rightarrow \quad \text{not a problem}$$

- logarithms of zero or negative numbers

$$f(x) = \log_2(x) \quad \rightarrow \quad x > 0$$

$$f(x) = \log_3(x) \quad \rightarrow \quad x > 0$$

Task 1: Find the natural domain of

$$f(x) = \frac{\sqrt{x+9}}{5x^2 + 10x - 40}.$$

Task 2: Find the natural domain of

$$f(x) = \sqrt{x^2 - 9}.$$

Combining functions

Given two functions $f(x)$ and $g(x)$, we can create the

- **sum** $f(x) + g(x)$,
- **difference** $f(x) - g(x)$,
- **product** $f(x) \cdot g(x)$,
- **quotient** $\frac{f(x)}{g(x)}$, and
- **composition** $f(g(x))$.

The first four words are also used for numbers (e.g., 5 is the sum of 2 and 3), but composition is only used for functions.

Question: Is

$$x^5 \sin(x^3 + 1) - 8$$

a sum, product, or composition?

A diagram can help us understand a function more carefully.

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

we can find the output-value for a specific input-value. For this example,

$$f(1) = \frac{1^2 - 1 - 2}{1 - 2} = \frac{-2}{-1} = 2.$$

What about when $x = 2$?

$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0} = ☹️.$$

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

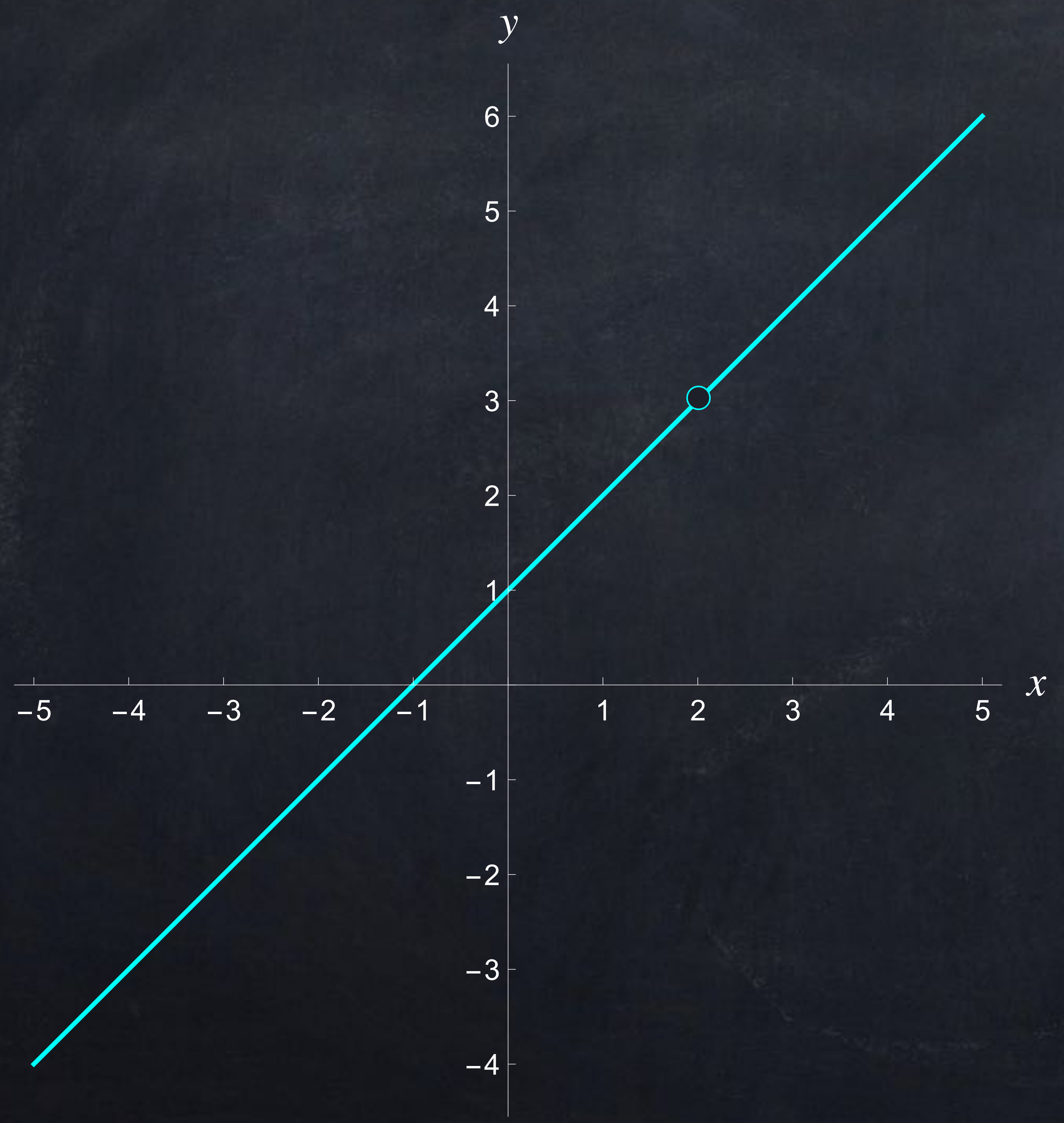
when $x = 2$,

$$f(2) = \frac{2^2 - 2 - 2}{2 - 2} = \frac{0}{0} = 🙁.$$

But if we look at the graph $y = \dfrac{x^2 - x - 2}{x - 2}$, we will be able to say more about $f(2)$.

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$



All of the x-values very close to 2 give us values of $f(x)$ very close to 3.

In symbols, we write

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = 3.$$

For the function

$$f(x) = \frac{x^2 - x - 2}{x - 2},$$

we can also use a table of values to find $\lim_{x \to 2} f(x)$.

x	1.8	1.9	1.99	1.999	2.001	2.005	2.1
$f(x)$	2.8	2.9	2.99	2.999	3.001	3.005	3.1

These are very close to 3.

Note: this "limit" is about what happens when the input is CLOSE to a certain value but NOT exactly equal to it. We do NOT include x = 2 in this table.

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

The equation above is said out loud as

"the limit as X goes to A of F of X equals L"

or

"the limit as X approaches A of F of X equals L".

Limits as x → a

In general, we write

$$\lim_{x \to a} f(x) = L,$$

if all values of x very close a give values of $f(x)$ that are very close to L.

There is an official definition using "ε" as any small value:

- $\lim_{x \to a} f(x) = L$ means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

  if $a-\delta < x < a+\delta$ then $L-\varepsilon < f(x) < L+\varepsilon$.

Example: find $\lim_{x \to 5} \frac{x-5}{x^2-25}$.

Method 1: table

x	4.9	4.95	4.99	4.999	5.001	5.005	5.02	5.1
$f(x)$								

Method 2: graph

Method 3: algebra